TMA4170 Fourier Analysis

Fourier transform in $C_m(\mathbb{R}^d)$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^d$$

Well-defined for

$$f \in C_m(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) : \sup_{x \in \mathbb{R}} |x|^{d+\varepsilon} |f(x)| < \infty, \varepsilon > 0 \}$$

Ex: e^{-x^2}, $\frac{1}{1+|x|^{d+\varepsilon}}$

Proposition 1: Properties for $f \in C_m(\mathbb{R}^d)$ (or $L^1(\mathbb{R}^d)$)

(a) $\hat{f} \in C_0(\mathbb{R}^d)$, $\|\hat{f}\|_\infty \le \|f\|_1$, $|\hat{f}(\xi)| \xrightarrow[|\xi| \to \infty]{} 0$

(b) $\mathcal{F}[f(x+h)](\xi) = \hat{f}(\xi)\, e^{2\pi i h\cdot\xi}$

(c) $\mathcal{F}[f(x) e^{-2\pi i x\cdot h}](\xi) = \hat{f}(\xi + h)$

(d) $\mathcal{F}[f(\delta x)](\xi) = \delta^{-d} \hat{f}(\delta^{-1}\xi)$, $\delta > 0$

(e) $\mathcal{F}[f(Rx)](\xi) = \hat{f}(R\xi)$ $\forall$ rotations R

Radial functions:

$f(x)$ radial if $\exists f_0 : [0,\infty) \to \mathbb{C}$ such that $f(x) = f_0(|x|) \ \forall x \in \mathbb{R}^d$

f radial $\iff$ $f(Rx) = f(x) \ \forall$ rotations R, $x \in \mathbb{R}^d$

Proposition 2: $f \in C_m(\mathbb{R}^d)$ radial $\implies$ $\hat{f}$ radial